

# ALGEBRAIC $K$ -THEORY WITH COEFFICIENTS OF CYCLIC QUOTIENT SINGULARITIES

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ABSTRACT. In this short note, by combining the work of Amiot-Iyama-Reiten and Thanhoffer de Völsey-Van den Bergh on Cohen-Macaulay modules with the previous work of the author on orbit categories, we compute the (nonconnective) algebraic  $K$ -theory with coefficients of cyclic quotient singularities.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $k$  be an algebraically closed field of characteristic zero. Given an integer  $d \geq 2$ , consider the associated polynomial ring  $S := k[t_1, \dots, t_d]$ . Let  $G$  be a cyclic subgroup of  $\mathrm{SL}(d, k)$  generated by  $\mathrm{diag}(\zeta^{a_1}, \dots, \zeta^{a_d})$ , where  $\zeta$  is a primitive  $n^{\mathrm{th}}$  root of unit and  $a_1, \dots, a_d$  are integers satisfying the following conditions: we have  $0 < a_j < n$  and  $\gcd(a_j, n) = 1$  for every  $1 \leq j \leq d$ ; we have  $a_1 + \dots + a_d = n$ . The group  $G$  acts naturally on  $S$  and the invariant ring  $R := S^G$  is a Gorenstein isolated singularity of Krull dimension  $d$ . For example, when  $d = 2$ , the ring  $R$  identifies with the Kleinian singularity  $k[u, v, w]/(u^n + vw)$  of type  $A_{n-1}$ .

The affine  $k$ -scheme  $X := \mathrm{Spec}(R)$  is singular. Following Orlov [3, 4], we can then consider the associated dg category of singularities  $\mathcal{D}_{\mathrm{dg}}^{\mathrm{sing}}(X)$ ; also known as matrix factorizations or maximal Cohen-Macaulay modules. Roughly speaking, this dg category encodes all the crucial information concerning the isolated singularity of  $X$ .

Let us denote by  $(Q, \rho)$  the quiver with relations defined by the following steps:

- (s1) consider the quiver with vertices  $\mathbb{Z}/n\mathbb{Z}$  and with arrows  $x_j^i: i \rightarrow i + a_j$ , where  $i \in \mathbb{Z}/n\mathbb{Z}$  and  $1 \leq j \leq d$ . The relations  $\rho$  are given by  $x_{j'}^{i+a_j} x_j^i = x_j^{i+a_{j'}} x_{j'}^i$  for every  $i \in \mathbb{Z}/n\mathbb{Z}$  and  $1 \leq j, j' \leq d$ .
- (s2) remove from (s1) all arrows  $x_j^i: i \rightarrow i'$  with  $i > i'$ ;
- (s3) remove from (s2) the vertex 0.

Consider the matrix  $(n-1) \times (n-1)$  matrix  $C$  such that  $C_{ij}$  equals the number of arrows in  $Q$  from  $j$  to  $i$  (counted modulo the relations). Let us write  $M$  for the matrix  $(-1)^{d-1} C(C^{-1})^T - \mathrm{Id}$  and  $M: \bigoplus_{r=1}^{n-1} \mathbb{Z}/l^\nu \rightarrow \bigoplus_{r=1}^{n-1} \mathbb{Z}/l^\nu$  for the associated (matrix) homomorphism, where  $l^\nu$  is a (fixed) prime power.

**Theorem 1.1.** *We have the following computation:*

$$\mathcal{K}_i(\mathcal{D}_{\mathrm{dg}}^{\mathrm{sing}}(X); \mathbb{Z}/l^\nu) \simeq \begin{cases} \text{cokernel of } M & \text{if } i \geq 0 \text{ even} \\ \text{kernel of } M & \text{if } i \geq 0 \text{ odd} \\ 0 & \text{if } i < 0. \end{cases}$$

Thanks to Theorem 1.1, the computation of the (nonconnective) algebraic  $K$ -theory with coefficients of the cyclic quotient singularities reduces to the computation of (co)kernels of explicit matrix homomorphisms! To the best of the author's

*Date:* December 4, 2015.

The author was partially supported by a NSF CAREER Award.

knowledge, these computations are new in the literature. In the particular case of Kleinian singularities of type  $A_n$  they were originally established in [7, §3]

**Corollary 1.2.** (i) *If there exists a prime power  $l^\nu$  and an even (resp. odd) integer  $j \geq 0$  such that  $\mathbb{K}_j(\mathcal{D}_{\text{dg}}^{\text{sing}}(X); \mathbb{Z}/l^\nu) \neq 0$ , then for every even (resp. odd) integer  $i \geq 0$  at least one of the groups  $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)), \mathbb{K}_{i-1}(\mathcal{D}_{\text{dg}}^{\text{sing}}(X))$  is non-zero.*

(ii) *If there exists a prime power  $l^\nu$  such that  $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X); \mathbb{Z}/l^\nu) = 0$  for every  $i \geq 0$ , then the groups  $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)), i \geq 0$ , are uniquely  $l^\nu$ -divisible.*

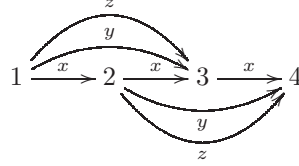
*Proof.* Combine the universal coefficients sequence (see [7, §5])

$$0 \longrightarrow \mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)) \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \longrightarrow \mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X); \mathbb{Z}/l^\nu) \longrightarrow {}_{l^\nu}\mathbb{K}_{i-1}(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)) \longrightarrow 0$$

with the computation of Theorem 1.1.  $\square$

## 2. EXAMPLES

**A low dimensional example.** When  $d = 3$ ,  $n = 5$ ,  $a_1 = 1$ , and  $a_2 = a_3 = 2$ , the above three steps (s1)-(s3) lead to the following quiver

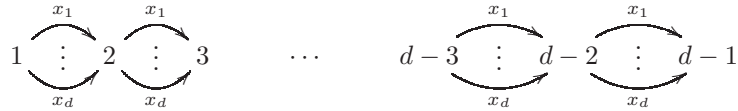


with relations  $xy = yx$ ,  $yz = zy$ , and  $zx = xz$ . Consequently, we obtain the matrix

$$M = \begin{pmatrix} 0 & -1 & -3 & -3 \\ 1 & -1 & -4 & -6 \\ 3 & -2 & -10 & -13 \\ 3 & 0 & -11 & -19 \end{pmatrix}.$$

Since  $\det(M) = 26$ , we have  $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X); \mathbb{Z}/l^\nu) = 0$  whenever  $l \neq 2, 13$ . In the remaining two cases, a computation shows that  $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X); \mathbb{Z}/l^\nu) \simeq \mathbb{Z}/l$  for every  $i \geq 0$ . Thanks to Corollary 1.2, this implies that for every  $i \geq 0$  at least one of the groups  $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)), \mathbb{K}_{i-1}(\mathcal{D}_{\text{dg}}^{\text{sing}}(X))$  is non-trivial. Moreover, the groups  $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X)), i \geq 0$ , are uniquely  $l$ -divisible for every prime  $l \neq 2, 13$ .

**A family of examples.** When  $n = d \geq 3$  and  $a_1 = \dots = a_d = 1$ , the above three steps (s1)-(s3) lead to the following quiver



with relations  $x_j x_i = x_i x_j$ . In the case where  $d$  is odd, we obtain the matrix

$$M_{ij} = \begin{cases} -\sum_{r=0}^{i-1} \binom{d}{r} \binom{d}{(j-i)+r} & \text{if } i < j \\ -\sum_{r=1}^{i-1} \binom{d}{r}^2 & \text{if } i = j \\ -\sum_{r=1}^{j-1} \binom{d}{(i-j)+r} \binom{d}{r} + \binom{d}{i-j} & \text{if } i > j, \end{cases}$$

where  $\left(\!\!\left(\!\!\right)\!\!\right)$  stands for the multicomination<sup>1</sup> symbol. Similarly, in the case where  $d$  is even, we obtain the matrix

$$M_{ij} = \begin{cases} \sum_{r=0}^{i-1} \left(\!\!\left(\!\!\right)\!\!\right) \left(\!\!\left(\!\!\right)\!\!\right)_{(j-i)+r}^d & \text{if } i < j \\ -2 + \sum_{r=1}^{i-1} \left(\!\!\left(\!\!\right)\!\!\right)_r^d & \text{if } i = j \\ \sum_{r=1}^{j-1} \left(\!\!\left(\!\!\right)\!\!\right)_{(i-j)+r}^d \left(\!\!\left(\!\!\right)\!\!\right)_r^d - \left(\!\!\left(\!\!\right)\!\!\right)_{i-j}^d & \text{if } i > j. \end{cases}$$

Whenever  $d$  is a prime number, all the multicombinations

$$\left(\!\!\left(\!\!\right)\!\!\right)_r^d = \binom{d+r-1}{r} = \frac{(d+r-1) \cdots d(d-1)!}{r!(d-1)!} \quad 0 \leq r \leq d-2$$

are multiples of  $d$ . This implies that the homomorphism  $M: \oplus_{r=1}^{d-1} \mathbb{Z}/d \rightarrow \oplus_{r=1}^{d-1} \mathbb{Z}/d$  is zero, and consequently that  $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X); \mathbb{Z}/d) \simeq \oplus_{r=1}^{d-1} \mathbb{Z}/d$  for every  $i \geq 0$ . These isomorphisms are a far reaching generalization of the particular case  $d = 3$  originally established in [7, Prop. 3.4]. Thanks to Corollary 1.2(i), we hence conclude that for every  $i \geq 0$  at least one of the groups  $\mathbb{K}_i(\mathcal{D}_{\text{dg}}^{\text{sing}}(X))$ ,  $\mathbb{K}_{i-1}(\mathcal{D}_{\text{dg}}^{\text{sing}}(X))$  is non-trivial.

### 3. PROOF OF THEOREM 1.1

Let  $A$  be a finite dimensional  $k$ -algebra of finite global dimension. We write  $\mathcal{D}^b(A)$  for the bounded derived category of (right)  $A$ -modules and  $\mathcal{D}_{\text{dg}}^b(A)$  for the canonical dg enhancement of  $\mathcal{D}^b(A)$ . Consider the following dg functors

$$\tau^{-1}\Sigma^d: \mathcal{D}_{\text{dg}}^b(A) \longrightarrow \mathcal{D}_{\text{dg}}^b(A) \quad d \geq 0,$$

where  $\tau$  stands for the Auslander-Reiten translation. Following Keller [2, §7.2], we can consider the associated dg orbit category  $\mathcal{C}_A^{(d)} := \mathcal{D}_{\text{dg}}^b(A)/(\tau^{-1}\Sigma^d)^{\mathbb{Z}}$ . Similarly to [7, Thm. 2.5] (consult [6, §2]), we have a distinguished triangle of spectra

$$\bigoplus_{r=1}^v \mathbb{K}(k; \mathbb{Z}/l^\nu) \xrightarrow{(-1)^d \Phi_A - \text{Id}} \bigoplus_{r=1}^v \mathbb{K}(k; \mathbb{Z}/l^\nu) \rightarrow \mathbb{K}(\mathcal{C}_A^{(d)}; \mathbb{Z}/l^\nu) \rightarrow \bigoplus_{r=1}^v \Sigma \mathbb{K}(k; \mathbb{Z}/l^\nu),$$

where  $v$  stands for the number of simple (right)  $A$ -modules and  $\Phi_A$  for the inverse of the Coxeter matrix of  $A$ . Consider the (matrix) homomorphism

$$(3.1) \quad (-1)^d \Phi_A - \text{Id}: \bigoplus_{r=1}^v \mathbb{Z}/l^\nu \longrightarrow \bigoplus_{r=1}^v \mathbb{Z}/l^\nu.$$

As proved by Suslin in [5, Cor. 3.13], we have  $\mathbb{K}_i(k; \mathbb{Z}/l^\nu) \simeq \mathbb{Z}/l^\nu$  when  $i \geq 0$  is even and  $\mathbb{K}_i(k; \mathbb{Z}/l^\nu) = 0$  otherwise. Consequently, making use of the long exact sequence of algebraic  $K$ -theory groups with coefficients associated to the above distinguished triangle of spectra, we obtain the following computations:

$$\mathbb{K}_i(\mathcal{C}_A^{(d)}; \mathbb{Z}/l^\nu) \simeq \begin{cases} \text{cokernel of (3.1)} & \text{if } i \geq 0 \text{ even} \\ \text{kernel of (3.1)} & \text{if } i \geq 0 \text{ odd} \\ 0 & \text{if } i < 0. \end{cases}$$

Consider also the following dg functors

$$S^{-1}\Sigma^d: \mathcal{D}_{\text{dg}}^b(A) \longrightarrow \mathcal{D}_{\text{dg}}^b(A) \quad d \geq 0,$$

<sup>1</sup>Also known as the *multisubset* symbol.

where  $S$  stands for the Serre dg functor. The associated dg orbit category  $\mathcal{C}_d(A) := \mathcal{D}_{\text{dg}}^b(A)/(S^{-1}\Sigma^d)^{\mathbb{Z}}$  is usually called the *generalized  $d$ -cluster dg category of  $A$* ; see [1, §1.3] and the references therein. Since  $S^{-1}\Sigma = \tau^{-1}$ , we have  $\mathcal{C}_d(A) \simeq \mathcal{C}_A^{(d-1)}$ .

Now, let us take for  $A$  the  $k$ -algebra  $kQ/\langle \rho \rangle$  associated to the quiver with relations  $(Q, \rho)$ . As proved independently by Amiot-Iyama-Reiten [1, §5] and Thanhoffer de Völcsey-Van den Bergh [8], we have  $\mathcal{D}_{\text{dg}}^{\text{sing}}(X) \simeq \mathcal{C}_{d-1}(A)$ . Consequently, it remains then only to show that the homomorphism (3.1), with  $d$  replaced by  $d-2$ , agrees with the homomorphism  $M$  associated to the matrix  $M := (-1)^{d-1}C(C^{-1})^T - \text{Id}$ . On the one hand, the number of simple (right)  $A$ -modules agrees with the number of vertices of the quiver  $Q$ . This implies that  $v = n-1$ . On the other hand, the inverse of the Coxeter matrix of  $A$  can be expressed as  $-C(C^{-1})^T$ , where  $C_{ij}$  equals the number of arrows in  $Q$  from  $j$  to  $i$  (counted modulo the relations). This implies that  $(-1)^{d-2}\Phi_A - \text{Id} = (-1)^{d-1}C(C^{-1})^T - \text{Id} = M$ , and hence concludes the proof.

**Acknowledgments:** The author is grateful to Michel Van den Bergh for useful discussions.

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